

COMMUTING MATRICES AND THE HILBERT SCHEME OF POINTS ON AFFINE SPACES

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ABSTRACT. We give a linear algebraic description of the Hilbert scheme of points on the affine space of dimension n which naturally extends Nakajima's representation of the Hilbert scheme of points on the plane. We also introduce extended monads and perfect extended monads in order to generalize the monadic description of ideal sheaves of 0-dimensional subschemes of projective spaces. As an application of our ideas, we use results from the literature on commuting matrices to show the Hilbert scheme of c points on (\mathbb{C}^3) is irreducible for $c \leq 10$ and reducible for $c \geq 30$.

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1. INTRODUCTION

The Hilbert scheme $\text{Hilb}^{[c]}(\mathbb{C}^n)$ of c points in the affine space of dimension n parametrizes 0-dimensional subschemes of \mathbb{C}^d of length c . The case of $n = 2$ is much studied, though less is known about the higher dimensional cases.

The linear algebraic description of the $n = 2$ case given by Nakajima in [13, Chapter 1] is particularly relevant to us. One of the goals of this paper is to give a linear algebraic description of the Hilbert scheme $\text{Hilb}^{[c]}(\mathbb{C}^n)$ of c points on \mathbb{C}^n , which naturally extends Nakajima's representation of the punctual Hilbert scheme $\text{Hilb}^{[c]}(\mathbb{C}^2)$ of c points on \mathbb{C}^2 given in [13, Chapter 1]. This goal is attained in Sections 2 and 3.

More precisely, let V and W be complex vector spaces of dimension c and 1, respectively. Let B_1, \dots, B_n be operators on V commuting with each other and consider a map $I : W \rightarrow V$. Such data can be regarded as a point in the affine

variety $\mathcal{C}(n, c) \times \text{Hom}(W, V)$, where $\mathcal{C}(n, c)$ denotes the variety of n commuting $c \times c$ matrices. Our first result is to show that there is a set-theoretical bijection between $\text{Hilb}^{[c]}(\mathbb{C}^n)$ and the GIT quotient of $\mathcal{C}(n, c) \times \text{Hom}(W, V)$ modulo the natural action of $GL(V)$.

Another key ingredient in Nakajima's description is the relation between $\text{Hilb}^{[c]}(\mathbb{C}^2)$ and linear monads on the projective space \mathbb{P}^2 . In Section 4 we introduce *extended monads*, and show in Section 5 that for every ideal sheaf \mathcal{I}_Z of a 0-dimensional subscheme $Z \subset \mathbb{P}^n$ there is a complex, called a *perfect extended monad*, of the form

$$\begin{aligned} P^\bullet : \quad \mathcal{O}_{\mathbb{P}^n}(1-n)^{\oplus a_1-n} &\xrightarrow{\alpha_{1-n}} \mathcal{O}_{\mathbb{P}^n}(2-n)^{\oplus a_2-n} \longrightarrow \\ &\cdots \xrightarrow{\alpha_{-2}} \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus a_{-1}} \xrightarrow{\alpha_{-1}} \mathcal{O}_{\mathbb{P}^n}^{\oplus a_0} \xrightarrow{\alpha_0} \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus a_1} \end{aligned}$$

which is exact everywhere except at 0, and $\mathcal{H}^0(P^\bullet) = \mathcal{I}_Z$ (grading of the complex is given by the twisting). This monadic description is connected back to the linear algebraic description in Section 6, where we provide a 1-1 correspondence between perfect extended monads and commuting matrices.

The monadic description is also important to show that the set-theoretical bijection $\text{Hilb}^{[c]}(\mathbb{C}^n)$ and the GIT quotient $\mathcal{C}(n, c) \times \text{Hom}(W, V) // GL(V)$ is actually a scheme theoretic isomorphism, as established in Section 7. In Section 8, we provide a description of the Hilbert–Chow morphism from $\text{Hilb}^{[c]}(\mathbb{C}^n)$ to the symmetric product of c copies of \mathbb{C}^n in terms of our linear data.

As an application of our ideas, we show in Section 9 that $\text{Hilb}^{[c]}(\mathbb{C}^n)$ and $\mathcal{C}(n, c)$ have the same number of irreducible components. We then use known results in the literature of commuting matrices to deduce new interesting facts on the irreducibility of $\text{Hilb}^{[c]}(\mathbb{C}^3)$: previously, it was known that $\text{Hilb}^{[c]}(\mathbb{C}^3)$ is irreducible if $c \leq 8$ and reducible for $c \geq 78$ [10] (see also [1, Section 7]); we improve these bounds by showing that $\text{Hilb}^{[c]}(\mathbb{C}^3)$ is irreducible if $c \leq 10$ and reducible for $c \geq 30$.

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2. COMMUTING MATRICES AND STABLE ADHM DATA

In this section we shall introduce the necessary material to our construction: let V be a complex vector space of dimension c and let $B_0, B_1, \dots, B_{n-1} \in \text{End}(V)$ be n linear operators on V .

Definition 2.1. *The variety $\mathcal{C}(n, c)$ of n commuting linear operators on V is the subvariety of $\text{End}(V)^{\oplus n}$ whose points are the set of n -tuples $(B_0, B_1, \dots, B_{n-1})$ that commutes two by two, that is,*

$$\mathcal{C}(n, c) = \{(B_0, B_1, \dots, B_{n-1}) \in \text{End}(V)^{\oplus n} \mid [B_i, B_j] = 0, \forall i \neq j\}$$

The commutation relations can be thought of as a system of $\binom{n}{2}c^2$ homogeneous equations of degree 2 in nc^2 variables.

Let W be a 1-dimensional complex vector space; one can form the space

$$\mathbb{B} := \text{End}(V)^{\oplus n} \oplus \text{Hom}(W, V)$$

whose points are represented by the $(n+1)$ -tuple $X = (B_0, B_1, \dots, B_{n-1}, I)$ that will be called an *ADHM datum*. We then define the *variety of ADHM data* $\mathcal{V}(n, c)$ as the subvariety of \mathbb{B} given by

$$\mathcal{V}(n, c) := \mathcal{C}(n, c) \times \text{Hom}(W, V).$$

Definition 2.2. An ADHM datum $X = (B_0, B_1, \dots, B_{n-1}, I) \in \mathbb{B}$ is said to be stable if there is no proper subspace $S \subsetneq V$ such that

$$B_0(S), B_1(S), \dots, B_{n-1}(S), I(W) \subset S.$$

The set of stable points in \mathbb{B} will be denoted by \mathbb{B}^{st} ; $\mathcal{V}(n, c)^{st} := \mathbb{B}^{st} \cap \mathcal{V}(n, c)$ will denote the set of stable points in $\mathcal{V}(n, c)$.

Definition 2.3. The stabilizing subspace Σ_X of an ADHM datum $X \in \mathbb{B}$ is the intersection of all subspaces $S \subset V$ such that $B_0(S), B_1(S), \dots, B_{n-1}(S), I(W) \subset S$.

It is easy to see that X is stable if and only if $\Sigma_X = V$. The restricted ADHM datum $X|_{\Sigma_X} = (B_0|_{\Sigma_X}, B_1|_{\Sigma_X}, \dots, B_{n-1}|_{\Sigma_X}, I|_{\Sigma_X})$ is stable in $\mathbb{B}|_{\Sigma_X} = \text{End}(\Sigma_X)^{\oplus n} \oplus \text{Hom}(W, \Sigma_X)$. The space $\Sigma_X \subset V$ is the smallest subspace which makes the datum $X|_{\Sigma_X}$ stable, hence the name.

For each ADHM datum $X = (B_0, \dots, B_{n-1}, I) \in \mathbb{B}$, we consider the linear map $\mathcal{R}_n(X) : W^{\oplus c^n} \rightarrow V$ defined by

$$\begin{aligned} \mathcal{R}_n(X) : W^{\oplus c^n} &\longrightarrow V \\ \bigoplus_{k_0, \dots, k_{n-1}=0}^{c-1} w_{k_0, \dots, k_{n-1}} &\longmapsto \sum_{k_0, \dots, k_{n-1}=0}^{c-1} B_0^{k_0} B_1^{k_1} \dots B_{n-1}^{k_{n-1}} I w_{k_0, \dots, k_{n-1}} \end{aligned}$$

One might think of \mathcal{R}_n as a regular morphism $\mathbb{B} \rightarrow \text{Hom}(W^{\oplus c^n}, V)$, hence continuous in the Zariski topology.

Proposition 2.4. For every $X \in \mathbb{B}$ one has

- (1) $\text{Im } \mathcal{R}_n(X) \subseteq \Sigma_X$;
- (2) if $\mathcal{R}_n(X)$ is surjective, then X is stable.

Proof. For any $S \subseteq V$ satisfying $B_0(S), B_1(S), \dots, B_{n-1}(S), I(W) \subset S$ we have $\text{Im } \mathcal{R}_n(X) \subseteq S$ which in particular implies our first assertion.

Moreover, if $\mathcal{R}_n(X)$ is surjective then we have $c = \text{rk } \mathcal{R}_n(X) \leq \dim \Sigma_X \leq c$. Therefore $\dim \Sigma_X = c$ and X is stable. \square

If the ADHM datum X is in $\mathcal{V}(n, c)$, then we obtain the following stronger characterization.

Proposition 2.5. For every datum $X = (B_0, \dots, B_{n-1}, I) \in \mathcal{V}(n, c)$ one has:

- (1) $\text{Im } \mathcal{R}_n(X) = \Sigma_X$.
- (2) $\mathcal{R}_n(X)$ is surjective if and only if X is stable.

Proof. Note first that the second claim follows easily from Proposition 2.4 and the first claim.

To prove the first claim, we only need to prove the inverse inclusion $\Sigma_X \subseteq \text{Im } \mathcal{R}_n(X)$ for those ADHM data which belong to $\mathcal{V}(n, c)$ since the inclusion $\text{Im } \mathcal{R}_n(X) \subseteq \Sigma_X$ holds for all $X \in \mathbb{B}$.

For this end, it is enough show that $\text{Im } \mathcal{R}_n(X)$ is B_i -invariant, for all $i \in \{0, \dots, n-1\}$, and $I(W) \subseteq \text{Im } \mathcal{R}_n(X)$. It is easy to see that $I(W) \subseteq \text{Im } \mathcal{R}_n(X)$, so

the results follows by showing that $\text{Im } \mathcal{R}_n(X)$ is B_i -invariant, for all $i \in \{0, \dots, n-1\}$; let

$$\sum_{k_0, \dots, k_{n-1}=0}^{c-1} B_0^{k_0} B_1^{k_1} \dots B_{n-1}^{k_{n-1}} Iw_{k_0, \dots, k_{n-1}} \in \text{Im } \mathcal{R}_n(X).$$

For $X \in \mathcal{V}(n, c)$ one has the following identity

$$\begin{aligned} B_i \left(\sum_{k_0, \dots, k_{n-1}=0}^{c-1} B_0^{k_0} B_1^{k_1} \dots B_{n-1}^{k_{n-1}} Iw_{k_0, \dots, k_{n-1}} \right) &= \\ &= \sum_{k_0, \dots, \widehat{k_i}, \dots, k_{n-1}=0}^{c-1} B_0^{k_0} \dots B_i^c \dots B_{n-1}^{k_{n-1}} Iw_{k_0, \dots, k_{n-1}} \\ &\quad + \sum_{k_0, \dots, \widehat{k_i}, \dots, k_0=0}^{c-1} \sum_{k_i=1}^{c-1} B_0^{k_0} \dots B_i^{k_i} \dots B_{n-1}^{k_{n-1}} Iw_{k_0, \dots, k_{n-1}}. \end{aligned}$$

By the symbol $\widehat{}$ we mean omitting the term below it from the expression. It is clear that the second factor of the sum, in the lower line of the expression above, belongs to $\text{Im } \mathcal{R}_n(X)$. To see that also the first factor belongs to $\text{Im } \mathcal{R}_n(X)$, notice that the characteristic polynomial of B_i is of the form $p(x) = x^c + a_{c-1}x^{c-1} + \dots + a_1x + a_0$. Hence, by Cayley-Hamilton Theorem, it follows that B_i^c is given by a linear combination of lower powers of B_i , i. e., $B_i^c = -(a_{c-1}B_i^{c-1} + \dots + a_1B_i + a_0\mathbb{I}_V)$. With this, we conclude that $\sum_{k_0, \dots, \widehat{k_i}, \dots, k_{n-1}=0}^{c-1} B_0^{k_0} \dots B_i^c \dots B_{n-1}^{k_{n-1}} Iw_{k_0, \dots, k_{n-1}} \in \text{Im } \mathcal{R}_n(X)$ which, in particular means, that $\text{Im } \mathcal{R}_n(X)$ is B_i -invariant. Finally, since Σ_X is the smallest subspace of V with this properties, it then follows that $\Sigma_X \subseteq \text{Im } \mathcal{R}_n(X)$. \square

Next, we introduce the action of the linear group $G := GL(V)$ on \mathbb{B} . For all $g \in G$ and $X = (B_0, \dots, B_{n-1}, I) \in \mathbb{B}$, this action is given by

$$g \cdot (B_0, \dots, B_{n-1}, I) = (gB_0g^{-1}, \dots, gB_{n-1}g^{-1}, gI).$$

For a fixed ADHM datum X , we will denote by G_X its stabilizer subgroup:

$$G_X := \{g \in G \mid gX = X\} \subseteq G.$$

It is easy to see that X is stable if and only if gX is stable, and that G acts on $\mathcal{V}(n, c)$.

We conclude this section with two results relating stability in the sense of Definition 2.2 with GIT stability.

Proposition 2.6. *If $X \in \mathcal{V}(n, c)^{st}$, then its stabilizer subgroup G_X is trivial.*

Proof. Let $X = (B_0, \dots, B_{n-1}, I)$ be a stable ADHM datum and suppose that there exists an element $g \neq \mathbf{1}$ in G such that $gI = I$ and $gB_i g^{-1} = B_i$ for all $i \in \{0, \dots, n-1\}$. Then $\ker(g - \mathbf{1})$ is B_i -invariant, for all $i \in \{0, \dots, n-1\}$, and

$\text{Im } I \subseteq \ker(g - 1)$. Since X is stable, then $\ker(g - 1) \subset V$ must be the trivial subspace. Hence g must be the identity. \square

Let $\Gamma(\mathcal{V}(n, c))$ be the ring of regular functions on $\mathcal{V}(n, c)$. Fix $l > 0$, and consider the group homomorphism $\chi : G \rightarrow \mathbb{C}^*$ given by $\chi(g) = (\det g)^l$. This can be used for the construction of a suitable linearization of the G -action on $\mathcal{V}(n, c)$, that is, to lift the action of G on $\mathcal{V}(n, c)$ to an action on $\mathcal{V}(n, c) \times \mathbb{C}$ as follows: $g \cdot (X, z) := (g \cdot X, \chi(g)^{-1}z)$ for any ADHM datum $X \in \mathcal{V}(n, c)$ and $z \in \mathbb{C}$. Then one can form the scheme

$$\mathcal{V}(n, c) //_{\chi} G := \text{Proj} \left(\bigoplus_{i \geq 0} \Gamma(\mathcal{V}(n, c))^{G, \chi^i} \right)$$

where

$$\Gamma(\mathcal{V}(n, c))^{G, \chi^i} := \{f \in \Gamma(\mathcal{V}(n, c)) \mid f(g \cdot X) = \chi(g)^{-i} \cdot f(X), \forall g \in G\}.$$

The scheme $\mathcal{V}(n, c) //_{\chi} G$ is projective over the ring $\Gamma(\mathcal{V}(n, c))^G$ and quasi-projective over \mathbb{C} .

Proposition 2.7. *The orbit $G \cdot (X, z)$ is closed, for $z \neq 0$, if and only if the ADHM datum $X \in \mathcal{V}(n, c)$ is a stable.*

Proof. First, suppose that the orbit $G \cdot (X, z)$ is not closed, then there is a 1-parameter subgroup $\lambda : \mathbb{C}^* \rightarrow G$ such that the limit $(L, w) := \lim_{t \rightarrow 0} \lambda(t) \cdot (X, z)$ exists but does not belong to the orbit $G \cdot (X, z)$. The existence of the limit (L, w) implies that $\det(\lambda(t)) = t^N$ for some $N \leq 0$. If $N = 0$ then $\lambda(t) = \mathbb{I}_V$, which contradicts the fact that the limit does not belong to the orbit $G \cdot (X, z)$. Thus $N < 0$. Now, take the weight decomposition $V = \bigoplus_m V(m)$ of the space V , with respect to λ . Then the existence of a limit implies that

$$B_0(V(m)), B_1(V(m)), \dots, B_{n-1}(V(m)) \subset \bigoplus_{n \geq m} V(m) \text{ and } I(W) \subset \bigoplus_{n \geq 0} V(m).$$

The space $S := \bigoplus_{n \geq 0} V(m) \in V$ is proper since $N < 0$. Moreover, one has $B_0(S), B_1(S), \dots, B_{n-1}(S) \subset S$ and $I(W) \subset S$. Hence X is not stable.

Conversely, suppose that the ADHM datum $X = (B_0, B_1, \dots, B_{n-1}, I)$ is not stable. Then there exists a subspace $S \subset V$ such that $B_0(S), B_1(S), \dots, B_{n-1}(S) \subset S$ and $I(W) \subset S$. Let $T \subset V$ be a subspace such that $V = S \oplus T$. With respect to this decomposition, one can write the linear maps B_i , for $0 \leq i \leq n-1$ and I , as follows

$$B^i = \begin{pmatrix} \star & \star \\ 0 & \star \end{pmatrix}, \text{ for } 0 \leq i \leq n-1 \quad I = \begin{pmatrix} \star \\ 0 \end{pmatrix}.$$

Now, define the 1-parameter subgroup as

$$\lambda(t) = \begin{pmatrix} \mathbb{I}_S & 0 \\ 0 & t^{-1} \mathbb{I}_T \end{pmatrix},$$

then

$$\lambda(t) B_i \lambda(t)^{-1} = \begin{pmatrix} \star & t \star \\ 0 & \star \end{pmatrix} \text{ and } \lambda(t) I = I.$$

It follows that the limit $L = \lim_{t \rightarrow 0} \lambda(t) \cdot X$ exists and $\lim_{t \rightarrow 0} \lambda(t) \cdot (X, z) = (L, 0)$, which means that the orbit is closed within $\mathcal{V}(n, c) \times \mathbb{C}^*$. \square

From Propositions 2.6 and 2.7 and since the group G is reductive, it follows that the quotient space $\mathcal{M}(n, c) := \mathcal{V}(n, c) //_{\chi} G$ is a good categorical quotient [12, Thm. 1.10]. Furthermore, GIT tells us that the GIT quotient $\mathcal{M}(n, c)$ is the space of orbits $G \cdot X \subset \mathcal{V}(n, c)$ such that the lifted orbit $G \cdot (X, z)$ is closed within $\mathcal{V}(n, c) \times \mathbb{C}$ for all $z \neq 0$. We conclude therefore, from Proposition 2.7, that

$$\mathcal{M}(n, c) = \mathcal{V}(n, c)^{\text{st}} / G.$$

3. PARAMETRIZATION OF THE HILBERT SCHEME OF c POINTS IN \mathbb{C}^n

As a set, the Hilbert scheme of c points on \mathbb{C}^n is given by:

$$\text{Hilb}^{[c]}(\mathbb{C}^n) = \{I \triangleleft \mathbb{C}[z_0, \dots, z_{n-1}] \mid \dim_{\mathbb{C}}(\mathbb{C}[z_0, \dots, z_{n-1}]/I) = c\}.$$

The existence of its schematic structure is a special case of the general result of Grothendieck [6]. Another explicit construction of the Hilbert scheme of points on the affine plane is given by Nakajima [13]. The reader may also consult [14] for more general results and examples.

The aim of this section is to prove the following result

Theorem 3.1. *There exists a set-theoretical bijection between the quotient space $\mathcal{M}(n, c)$ and the Hilbert scheme of c points in \mathbb{C}^n .*

We remark that this result will be strengthened, in Section 7 below, to a scheme theoretic isomorphism rather than just a bijective correspondence. Before proving the above result it will be useful to, first, establish a few lemmata.

Lemma 3.2. *If $X = (B_0, \dots, B_{n-1}, I) \in \mathcal{V}(n, c)^{\text{st}}$ is a stable ADHM datum, then the map:*

$$\begin{aligned} \Phi_X : \mathbb{C}[Z_0, \dots, Z_{n-1}] &\longrightarrow V \\ p(Z_0, \dots, Z_{n-1}) &\longmapsto p(B_0, \dots, B_{n-1})I(1) \end{aligned}$$

is a surjective linear transformation. In particular, $\mathbb{C}[Z_0, \dots, Z_{n-1}] / \ker \Phi_X$ is isomorphic to V .

We Remark that the linear map Φ_X is well-defined, since $[B_i, B_j] = 0$, for all $0 \leq i < j \leq n-1$.

Proof. Observe that $\text{Im } I \subseteq \text{Im } \Phi_X$ since the elements of $\text{Im } I$ consist of vectors of the form $\alpha I(1)$, for some constant α in \mathbb{C} . The inverse image of such an element is simply the constant polynomial α itself. Moreover, the image $\text{Im } \Phi_X$ of the map Φ_X is B_i -invariant, for all $0 \leq i \leq n-1$, since all the B_i 's commute. By stability of the ADHM datum X we must have $\text{Im } \Phi_X = V$, and hence Φ_X is surjective.

It is clear that $\ker \Phi_X \subset \mathbb{C}[Z_0, \dots, Z_{n-1}]$ is an ideal. Now, given any two polynomials $p(Z_0, \dots, Z_{n-1}) \in \mathbb{C}[Z_0, \dots, Z_{n-1}]$ and $q(Z_0, \dots, Z_{n-1}) \in \ker \Phi_X$, one has $\Phi_X(p(Z_0, \dots, Z_{n-1})q(Z_0, \dots, Z_{n-1})) = p(B_0, \dots, B_{n-1})q(B_0, \dots, B_{n-1})I(1) = 0$. Hence the isomorphism $\mathbb{C}[Z_0, \dots, Z_{n-1}] / \ker \Phi_X \simeq V$. \square

Let $\pi : \mathcal{V}(n, c)^{\text{st}} \rightarrow \mathcal{M}(n, c)$ be the natural projection onto the orbit space $\mathcal{M}(n, c)$ and denote by $[X] = [(B_0, \dots, B_{n-1}, I)]$ the class of the ADHM datum $X = (B_0, \dots, B_{n-1}, I)$ in $\mathcal{V}(n, c)^{\text{st}}$, that is, $[X] = \pi(X)$.

Lemma 3.3. *Let $X, Y \in \mathcal{V}(n, c)^{\text{st}}$ be two stable ADHM data such that $[X] = [Y]$. Then $\ker \Phi_X \simeq \ker \Phi_Y$.*

Proof. Suppose that $[X] = [(B_0, \dots, B_{n-1}, I)] = [Y] = [(A_0, \dots, A_{n-1}, J)]$, then there exists an element $g \in GL(V)$ such that $A_i = gB_i g^{-1}$, for all $0 \leq i \leq n-1$ and $J = gI$. Now, for any polynomial $f \in \mathbb{C}[Z_0, \dots, Z_{n-1}]$ one has $f(A_0, \dots, A_{n-1}) = gf(B_0, \dots, B_{n-1})g^{-1}$, hence

$$f(A_0, \dots, A_{n-1})J(1) = gf(B_0, \dots, B_{n-1})g^{-1}(gI(1)) = gf(B_0, \dots, B_{n-1})I(1),$$

in other words, $\Phi_Y = g\Phi_X$. Since g is invertible, it then follows that $\ker \Phi_X \simeq \ker \Phi_Y$. \square

Lemma 3.4. *The ADHM datum $X = (B_0, \dots, B_{n-1}, I) \in \mathcal{V}(n, c)$ is stable if and only if the set $\{B_0^{i_0} \cdot B_1^{i_1} \cdots B_{n-1}^{i_{n-1}} I(1) \in V \mid i_k = 0, \dots, c-1\}$ spans V as a complex vector space.*

Proof. The result follows from item (2) of Proposition 2.5 and the fact that the set $\{B_0^{i_0} \cdot B_1^{i_1} \cdots B_{n-1}^{i_{n-1}} I(1) \in V \mid i_k = 0, \dots, c-1\}$ spans $\text{Im } \mathcal{R}_n(X)$. \square

We are finally in position to complete the Proof of Theorem 3.1.

Proof of Theorem 3.1. We will consider the map

$$\begin{array}{ccc} \Psi : \mathcal{M}(n, c) & \longrightarrow & \text{Hilb}^{[c]}(\mathbb{C}^n) \\ [X] & \longmapsto & \ker \Phi_X \end{array}.$$

which associates the ideal $\ker \Phi_X$ to the class $[X] = [(B_0, \dots, B_{n-1}, I)]$ of a stable ADHM datum $X = (B_0, \dots, B_{n-1}, I) \in \mathcal{V}(n, c)^{st}$. By Lemma 3.3, the map Ψ is well-defined and it is clear from lemma 3.2 that $\ker \Phi_X$ belong to $\text{Hilb}^{[c]}(\mathbb{C}^n)$.

Inversely, we define the map

$$\begin{array}{ccc} \Psi' : \text{Hilb}^{[c]}(\mathbb{C}^n) & \longrightarrow & \mathcal{M}(n, c) \\ J & \longmapsto & [(B_0, \dots, B_{n-1}, I)] \end{array}$$

from $\text{Hilb}_{\mathbb{C}^n}^{[c]}$ to $\mathcal{M}(n, c)$ as the following:

Given an ideal $J \in \text{Hilb}^{[c]}(\mathbb{C}^n)$ we denote by $V = \mathbb{C}[Z_0, \dots, Z_{n-1}]/J$ the vector space associated to it. The multiplication by $Z_i \bmod J$ define endomorphisms $B_i \in \text{End}(V)$, for $0 \leq i \leq n-1$, in the following way

$$(1) \quad \begin{array}{ccc} B_i : V & \longrightarrow & V \\ [p(Z_0, \dots, Z_{n-1})] & \longmapsto & [Z_i p(Z_0, \dots, Z_{n-1})] \end{array}$$

One can also define $I \in \text{Hom}(W, V)$ as the linear mapping which associates to the unit vector $1 \in W$ the class $1 \bmod J \in V$. Since all B_i 's commute, then $(B_0^{i_0}, \dots, B_{n-1}^{i_{n-1}}, I) \in \mathcal{V}(n, c)$. Moreover, the set

$$\{B_0^{i_0} \cdot B_1^{i_1} \cdots B_{n-1}^{i_{n-1}} I(1) \in V \mid i_k = 0, \dots, c-1\}$$

spans V as complex vector space. Therefore, by Lemma 3.4, the ADHM datum X is stable.

To complete the proof, we only have to show that $\Psi' \circ \Psi = \mathbf{1}_{\mathcal{M}(n, c)}$ and $\Psi \circ \Psi' = \mathbf{1}_{\text{Hilb}_{\mathbb{C}^n}^{[c]}}$, i.e., the maps Ψ and Ψ' are inverse to each other.

Indeed, to each class $[X] \in \mathcal{M}(n, c)$, one associates the ideal $\Psi([X]) = \ker \Phi_X$ in $\text{Hilb}^{[c]}(\mathbb{C}^n)$. Moreover, one associates to the later ideal, $\ker \Phi_X$, the ADHM datum class $\Psi'(\ker \Phi_X) = [\tilde{X}]$. Then one has $[X] = [\tilde{X}]$ if and only if there exists an element $g \in GL(V)$ such that $\tilde{X} = g \cdot X$.

Let $pr : \mathbb{C}[Z_0, \dots, Z_{n-1}] \rightarrow \mathbb{C}[Z_0, \dots, Z_{n-1}] / \ker \Phi_X$ be the natural projection. From Lemma 3.2, it is clear that the diagram

$$\begin{array}{ccc} \mathbb{C}[Z_0, \dots, Z_{n-1}] & \xrightarrow{id} & \mathbb{C}[Z_0, \dots, Z_{n-1}] \\ pr \downarrow & & \downarrow \Phi_X \\ \mathbb{C}[Z_0, \dots, Z_{n-1}] / \ker \Phi_X & \xrightarrow{g} & V \end{array}$$

commutes. Hence $pr = g^{-1} \circ \Phi_X$, since g is an isomorphism. On the other hand, from the Z_i multiplication one has $pr \circ Z_i = \tilde{B}_i \circ pr : \mathbb{C}[Z_0, \dots, Z_{n-1}] \rightarrow \mathbb{C}[Z_0, \dots, Z_{n-1}] / \ker \Phi_X$ and $\Phi_X \circ Z_i = B_i \circ \Phi_X : \mathbb{C}[Z_0, \dots, Z_{n-1}] \rightarrow V$. That is, one has the following diagram

$$\begin{array}{ccccc} & & \mathbb{C}[Z_0, \dots, Z_{n-1}] & \xrightarrow{pr} & \mathbb{C}[Z_0, \dots, Z_{n-1}] / \ker \Phi_X \\ & \swarrow id & \downarrow \Phi_X & \swarrow g & \downarrow \tilde{B}_i \\ \mathbb{C}[Z_0, \dots, Z_{n-1}] & \xrightarrow{\quad} & \mathbb{C}^c & \xleftarrow{\quad} & \mathbb{C}[Z_0, \dots, Z_{n-1}] / \ker \Phi_X \\ \downarrow Z_i & & \downarrow B_i & & \downarrow \\ & \swarrow id & \mathbb{C}[Z_0, \dots, Z_{n-1}] & \xrightarrow{pr} & \mathbb{C}[Z_0, \dots, Z_{n-1}] / \ker \Phi_X \\ & \swarrow & \downarrow \Phi_X & \swarrow g & \\ \mathbb{C}[Z_0, \dots, Z_{n-1}] & \xrightarrow{\quad} & \mathbb{C}^c & \xleftarrow{\quad} & \mathbb{C}[Z_0, \dots, Z_{n-1}] / \ker \Phi_X \end{array}$$

in which all faces commute. Then, one has $g \circ \tilde{B}_i \circ g = B_i$, for all $i = \{0, \dots, n-1\}$. Moreover, $g \circ \tilde{I}(1) = g(1 \bmod J) = \Phi_X(1) = I(1)$, i. e., $g \circ \tilde{I} = I$. Therefore, $[X] = [\tilde{X}] \in \mathcal{M}(n, c)$, in other words, we have just shown that $\Psi' \circ \Psi = \mathbf{1}_{\mathcal{M}(n, c)}$.

To prove that $\Psi \circ \Psi' = \mathbf{1}_{\text{Hilb}_{\mathbb{C}^n}^{[c]}}$, we only need to show that for a given $J \in \text{Hilb}_{\mathbb{C}^n}^{[c]}$, one has $J = \ker \Phi_X$, where X is a ADHM datum in $\mathcal{V}(n, c)^{st}$ such that $\Psi'(J) = [X] \in \mathcal{M}(n, c)$. For a polynomial

$$p(Z_0, \dots, Z_{n-1}) = \sum_{\alpha} a_{\alpha} Z_0^{\alpha_0} \dots Z_{n-1}^{\alpha_{n-1}} \in \mathbb{C}[Z_0, \dots, Z_{n-1}]$$

we have

$$\Phi_X(p(Z_0, \dots, Z_{n-1})) = \sum_{\alpha} a_{\alpha} B_0^{\alpha_0} \dots B_{n-1}^{\alpha_{n-1}} I(1)$$

where $I(1)$ is the class $1 \bmod J =: [1]$. Moreover, since $B_i \circ \Phi_X = \Phi_X \circ Z_i$ then

$$\sum_{\alpha} a_{\alpha} B_0^{\alpha_0} \dots B_{n-1}^{\alpha_{n-1}} I(1) = \left[\sum_{\alpha} a_{\alpha} Z_0^{\alpha_0} \dots Z_{n-1}^{\alpha_{n-1}} \right] = [p(Z_0, \dots, Z_{n-1})].$$

Thus, if the polynomial $p(Z_0, \dots, Z_{n-1})$ belongs to the ideal J , then

$$\sum_{\alpha} a_{\alpha} B_0^{\alpha_0} \dots B_{n-1}^{\alpha_{n-1}} I(1) = 0,$$

and therefore $p(Z_0, \dots, Z_{n-1}) \in \ker \Phi_X$.

On the other hand, suppose that $p(Z_0, \dots, Z_{n-1}) \in \ker \Phi_X$. Then

$$\Phi_X(p(Z_0, \dots, Z_{n-1})) = \sum_{\alpha} a_{\alpha} B_0^{\alpha_0} \dots B_{n-1}^{\alpha_{n-1}} I(1) = 0.$$

Again, one has

$$\sum_{\alpha} a_{\alpha} B_0^{\alpha_0} \dots B_{n-1}^{\alpha_{n-1}} I(1) = \left[\sum_{\alpha} a_{\alpha} Z_0^{\alpha_0} \dots Z_{n-1}^{\alpha_{n-1}} \right] = [p(Z_0, \dots, Z_{n-1})],$$

hence $[p(Z_0, \dots, Z_{n-1})] = 0 \in \mathbb{C}[Z_0, \dots, Z_{n-1}]/J$, that is, $p(Z_0, \dots, Z_{n-1}) \in J$. Thus $J = \ker \Phi_X$. This finishes our proof. \square

4. EXTENDED MONADS AND PERFECT EXTENDED MONADS

In this section we shall generalize the concept of *monads*, introduced by Horrocks (the reader may consult [15] for definitions and properties), in order to describe ideal sheaves for zero-dimensional subschemes of \mathbb{C}^n and \mathbb{P}^n , $n \geq 2$.

Let X be a smooth projective algebraic variety of dimension n over the field of complex numbers \mathbb{C} , and let $\mathcal{O}_X(1)$ be a polarization on it.

4.1. l -extended monads. The objects we now wish to introduce are defined as follows.

Definition 4.1. *An l -extended monad over X is a complex*

$$(2) \quad C^{\bullet}: \quad C^{-l-1} \xrightarrow{\alpha_{-l-1}} C^{-l} \xrightarrow{\alpha_{-l}} \dots \xrightarrow{\alpha_{-2}} C^{-1} \xrightarrow{\alpha_{-1}} C^0 \xrightarrow{\alpha_0} C^1$$

of locally free sheaves over X which is exact at all but the 0-th position, i.e. $\mathcal{H}^i(C^{\bullet}) = 0$ for $i \neq 0$. The coherent sheaf $\mathcal{E} := \mathcal{H}^0(C^{\bullet}) = \ker \alpha_0 / \operatorname{Im} \alpha_{-1}$ will be called the cohomology of C^{\bullet} .

Note that a monad on X , in the usual sense, is just a 0-extended monad.

Moreover, one can associate to any l -extended monad C^\bullet a *display* of exact sequences as the following

$$(3) \quad \begin{array}{ccccccc} & 0 & & 0 & & & \\ & \downarrow & & \downarrow & & & \\ & C^{-l-1} = C_{-l-1} & & & & & \\ & \downarrow \alpha_{-l-1} & & \downarrow \alpha_{-l-1} & & & \\ & C^{-l} = C^{-l} & & & & & \\ & \downarrow \alpha_{-l} & & \downarrow \alpha_{-l} & & & \\ & \vdots & & \vdots & & & \\ & \downarrow \alpha_{-2} & & \downarrow \alpha_{-2} & & & \\ & C^{-1} = C^{-1} & & & & & \\ & \downarrow & & \downarrow \alpha_{-1} & & & \\ 0 \longrightarrow & K & \longrightarrow & C^0 & \xrightarrow{\alpha_0} & C^1 & \twoheadrightarrow 0 \\ & \downarrow & & \downarrow & & \parallel & \\ 0 \longrightarrow & \mathcal{F} & \longrightarrow & Q & \longrightarrow & C^1 & \twoheadrightarrow 0 \\ & \downarrow & & \downarrow & & & \\ & 0 & & 0 & & & \end{array}$$

where $K := \ker \alpha_0$ and $Q := \operatorname{coker} \alpha_{-1}$

A morphism $\phi : C_1^\bullet \rightarrow C_2^\bullet$ of two l -extended monads C_1^\bullet and C_2^\bullet is an $(l+3)$ -tuple of morphisms such that the following diagram commutes:

$$(4) \quad \begin{array}{ccccccc} C_1^\bullet : & C_1^{-l-1} \xrightarrow{\alpha_{-l-1}^1} & C_1^{-l} & \cdots \xrightarrow{\alpha_{-2}^1} & C_1^{-1} \xrightarrow{\alpha_{-1}^1} & C_1^0 \xrightarrow{\alpha_0^1} & C_1^1 \\ \downarrow \phi & \downarrow \phi_{-l-1} & \downarrow \phi_{-l} & & \downarrow \phi_{-1} & \downarrow \phi_0 & \downarrow \phi_1 \\ C_2^\bullet : & C_1^{-l-1} \xrightarrow{\alpha_{-l-1}^2} & C_1^{-l} & \cdots \xrightarrow{\alpha_{-2}^2} & C_1^{-1} \xrightarrow{\alpha_{-1}^2} & C_1^0 \xrightarrow{\alpha_0^2} & C_1^1 \end{array}$$

With these definitions, the category of l -extended monads form a full subcategory of the category $\operatorname{Kom}^b(X)$ of bounded complexes of coherent sheaves on X .

l -extended monads have already appeared in the literature. The most important example of a locally-free sheaf that can be obtained as the cohomology of a 2-extended monad on \mathbb{P}^4 is the dual of the Horrocks–Mumford bundle; indeed, Fløystad shows in [3, Introduction: example b.] that the Horrocks–Mumford bundle is given by the cohomology at degree zero, where the grading is given by the twist, of a complex of the form

$$\mathcal{O}_{\mathbb{P}^4}^5(-1) \rightarrow \mathcal{O}_{\mathbb{P}^4}^{15} \rightarrow \mathcal{O}_{\mathbb{P}^4}^{10}(1) \rightarrow \mathcal{O}_{\mathbb{P}^4}^2(2).$$

Dualizing such complex we get a 2-extended monad on \mathbb{P}^4 whose cohomology is the dual of the Horrocks–Mumford bundle.

Moreover, object very closely related to 2-extended monads on \mathbb{P}^3 have also appeared in the mathematical physics literature, see [2, Section 4].

An l -extended monad can be broken into the following two complexes: first,

$$(5) \quad N^\bullet : \quad 0 \longrightarrow C^{-l-1} \xrightarrow{\alpha_{-l-1}} C^{-l} \quad \dots \xrightarrow{\alpha_{-3}} C^{-2} \xrightarrow{\alpha_{-2}} C^{-1} \xrightarrow{J_{-1}} \mathcal{G} \longrightarrow 0$$

which is exact, and a locally free resolution of the sheaf $\mathcal{G} = \text{coker } \alpha_{-2}$, and

$$(6) \quad M^\bullet : \quad \mathcal{G} \xrightarrow{I_{-1}} C^0 \xrightarrow{\alpha_0} C^1$$

where $I_{-1} \circ J_{-1} = \alpha_{-1}$. M^\bullet is a monad-like complex in which the coherent sheaf \mathcal{G} might not be locally free; indeed, \mathcal{G} is not locally free for the extended monads describing ideal sheaves of 0-dimensional subschemes, the situation most relevant to the present paper.

For a given l -extended monad, we refer to the complexes M^\bullet and N^\bullet as the *associated resolution* and the *associated monad*, respectively. Therefore, the morphism $\phi : C_1^\bullet \rightarrow C_2^\bullet$ can be thought of as a pair of morphisms $(\phi_N : N_1^\bullet \rightarrow N_2^\bullet, \phi_M : M_1^\bullet \rightarrow M_2^\bullet) \in \text{Hom}(N_1^\bullet, N_2^\bullet) \times \text{Hom}(M_1^\bullet, M_2^\bullet)$.

Remark that as long as we have $\phi_0(\text{Im } \alpha_1^{-1}) \subseteq \text{Im } \alpha_2^{-1}$ and $\phi_0(\ker \alpha_0^1) \subseteq \ker \alpha_0^2$ then ϕ is determined by only ϕ_0 ; indeed, the conditions

$$\phi_0(\text{Im } \alpha_1^{-1}) \subseteq \text{Im } \alpha_2^{-1} \quad \text{and} \quad \phi_0(\text{Im } I_1^{-1}) \subseteq \text{Im } I_2^{-1}$$

are equivalent (here we considered the morphism of the associated monads). Hence ϕ_0 determines the morphism ϕ_M , and consequently it also determines the whole morphism $\phi : C_1^\bullet \rightarrow C_2^\bullet$. This is because N_1^\bullet and N_2^\bullet are locally free resolutions and hence projective resolutions, so that giving a morphism $\phi_G : \mathcal{G} \rightarrow \mathcal{G}$ determines all the morphisms $\phi_{-i} : C_1^{-i} \rightarrow C_2^{-i}$ for $i \leq -1$.

Since taking cohomology is a functorial operation, a morphism $\phi : C_1^\bullet \rightarrow C_2^\bullet$ of two l -extended monads C_1^\bullet and C_2^\bullet , induces a morphism between their respective cohomologies

$$H(\phi) : \mathcal{H}^0(C_1^\bullet) \rightarrow \mathcal{H}^0(C_2^\bullet).$$

Of course, isomorphic complexes induce isomorphic cohomologies. It follows that there is natural notion of equivalence for l -extended monads with the same terms C^i provided by the action of the automorphism group $\text{Aut}(C^\bullet) = \text{Aut}(C^{-l-1}) \times \text{Aut}(C^{-l}) \times \dots \times \text{Aut}(C^0) \times \text{Aut}(C^1)$.

Our goal now is to study families of ideal sheaves of zero-cycles in \mathbb{P}^n . It turns out that such ideal sheaves are given by cohomologies of a special kind of l -extended monads. However, before proving this claim, which will be done only in Section 5 below, we tackle a more general question, namely under which conditions a homomorphism $\mathcal{H}^0(C_1^\bullet) \rightarrow \mathcal{H}^0(C_2^\bullet)$ lifts to a homomorphism $C_1^\bullet \rightarrow C_2^\bullet$ between the corresponding complexes. In particular to determine the automorphisms of such objects.

Our next result provides a sufficient condition, by showing when the cohomology functor is full and faithful.

Proposition 4.2. *Let*

$$C_1^\bullet : \quad C_1^{-l-1} \xrightarrow{\alpha_{-l-1}^1} C_1^{-l} \quad \dots \xrightarrow{\alpha_{-2}^1} C_1^{-1} \xrightarrow{\alpha_{-1}^1} C_1^0 \xrightarrow{\alpha_0^1} C_1^1 \quad \text{and}$$

$$C_2^\bullet : \quad C_2^{-l-1} \xrightarrow{\alpha_{-l-1}^2} C_2^{-l} \quad \dots \xrightarrow{\alpha_{-2}^2} C_2^{-1} \xrightarrow{\alpha_{-1}^2} C_2^0 \xrightarrow{\alpha_0^2} C_2^1$$

be two l -extended monads, and let us denote by \mathcal{F}_1 and \mathcal{F}_2 their respective cohomologies. Then

$$H : \text{Hom}(C_1^\bullet, C_2^\bullet) \rightarrow \text{Hom}(\mathcal{F}_1, \mathcal{F}_2)$$

is surjective if

$$\text{Ext}^1(C_1^1, C_1^0) = 0,$$

$$\text{Ext}^k(C_1^0, C_2^{-k}) = 0 \text{ for } k \geq 1, \quad \text{Ext}^k(C_1^1, C_2^{-k+1}) = 0 \text{ for } k \geq 2.$$

Moreover if

$$\text{Hom}(C_1^1, C_2^0) = 0,$$

$$\text{Ext}^k(C_1^0, C_2^{-k+1}) = 0 \text{ for } k \geq 1, \quad \text{Ext}^k(C_1^0, C_2^{-k-1}) = 0 \text{ for all } k \geq 0,$$

then H is an isomorphism.

Proof. Let $\mathcal{G}_1 = \text{Im } \alpha_{-1}^1$ and $\mathcal{G}_2 = \text{coker } \alpha_{-1}^2$. The associated resolution N_2^\bullet can be broken into sequences

$$0 \rightarrow \mathcal{G}_2^{-i} \rightarrow C_2^{-i} \rightarrow \mathcal{G}_2^{-i+1} \rightarrow 0, \quad 1 \leq i \leq l+1$$

where we put $\mathcal{G}_2^{-l-1} = C_2^{-l}$ and $\mathcal{G}_2^0 = \mathcal{G}_2$. Then, by applying either $\text{Hom}(C_1^0, \bullet)$ or $\text{Hom}(C_1^1, \bullet)$ on the above sequences and incorporating the conditions given in the proposition, it follows that H is surjective if

$$\text{Ext}^1(C_1^1, C_1^0) = \text{Ext}^1(C_1^0, \mathcal{G}_2) = \text{Ext}^2(C_1^1, \mathcal{G}_2) = 0,$$

and it is an isomorphism if

$$\text{Hom}(C_1^1, C_2^0) = \text{Hom}(C_1^0, \mathcal{G}_2) = \text{Ext}^1(C_1^0, \mathcal{G}_2) = 0.$$

To finish the proof, it suffice to apply [15, Lemma 4.1.3] to the associated monad M_2^\bullet of the l -extended monad C_2^\bullet . \square

Corollary 4.3. *Let*

$$C^\bullet : \quad C^{-l-1} \xrightarrow{\alpha_{-l-1}} C^{-l} \quad \dots \xrightarrow{\alpha_{-2}} C^{-1} \xrightarrow{\alpha_{-1}} C^0 \xrightarrow{\alpha_0} C^1$$

and

$$C'^\bullet : \quad C^{-l-1} \xrightarrow{\alpha'_{-l-1}} C^{-l} \quad \dots \xrightarrow{\alpha'_{-2}} C^{-1} \xrightarrow{\alpha'_{-1}} C^0 \xrightarrow{\alpha'_0} C^1$$

be l -extended monads, and let \mathcal{F} and \mathcal{F}' be their cohomologies, respectively. Suppose that

$$\text{Ext}^1(C^1, C^0) = 0,$$

$$\text{Ext}^k(C^0, C^{-k}) = 0 \text{ for } k \geq 1, \quad \text{Ext}^k(C^1, C^{-k+1}) = 0 \text{ for } k \geq 2.$$

Then \mathcal{F} and \mathcal{F}' are isomorphic if and only if C^\bullet and C'^\bullet are isomorphic (as l -extended monads).

4.2. Perfect extended monads. We now introduce the class of l -extended monads which is relevant to the description of the Hilbert scheme of points.

Definition 4.4. An l -extended monad C^\bullet is called *pure* if $C^{-i} = \mathcal{L}_{-i}^{\oplus a-i}$, for all $-1 \leq i \leq l-1$, where \mathcal{L}_{-i} , are invertible sheaves, and it is called *linear* if all maps α_{-i} are given by matrices of linear forms.

Before our next definition, recall that $\mathcal{O}_X(1)$ is the chosen polarization on the n -dimensional projective algebraic variety X .

Definition 4.5. A perfect extended monad on a n -dimensional projective variety X is a linear $(n-2)$ -extended monad P^\bullet on X of the following form

$$\begin{aligned} \mathcal{O}_X(1-n)^{\oplus a_1-n} &\xrightarrow{\alpha_{1-n}} \mathcal{O}_X(2-n)^{\oplus a_2-n} \longrightarrow \\ &\dots \xrightarrow{\alpha_{-2}} \mathcal{O}_X(-1)^{\oplus a_{-1}} \xrightarrow{\alpha_{-1}} \mathcal{O}_X^{\oplus a_0} \xrightarrow{\alpha_0} \mathcal{O}_X(1)^{\oplus a_1}. \end{aligned}$$

We recall to the reader that a projective scheme X is *arithmetically Cohen-Macaulay*, or simply *ACM*, if its homogeneous coordinate ring is Cohen-Macaulay ring. Moreover let us denote by \mathfrak{Per} the full subcategory of $Kom^b(X)$ consisting of perfect extended monads.

Corollary 4.6. If X is an n -dimensional ACM variety, then the cohomology functor

$$H : \mathfrak{Per}(X) \rightarrow \text{Coh}(X)$$

is full and faithful.

Proof. This follows easily from Proposition 4.2: since X is ACM, we have that

$$\text{Hom}(C_1^1, C_2^0) = H^0(X, \mathcal{O}_X(-1)) = 0 \quad \text{and}$$

$$\text{Ext}^i(\mathcal{O}_X(a), \mathcal{O}_X(b)) = H^i(X, \mathcal{O}_X(b-a)) = 0, \text{ for } 1 \leq i \leq n-1.$$

□

It follows from the Corollary above that automorphism group of a perfect extended monad on an ACM variety is just $GL_{a_1-n}(\mathbb{C}) \times GL_{a_{-n}}(\mathbb{C}) \times \dots \times GL_{a_1}(\mathbb{C})$.

We finish this section by describing the cohomology of sheaves which arise as cohomologies of perfect extended monads on \mathbb{P}^n , $n \geq 2$.

Proposition 4.7. If \mathcal{F} is the cohomology of a perfect extended monad on \mathbb{P}^n ($n \geq 2$) then:

- (i) $H^0(\mathcal{F}(k)) = 0$ for $k < 0$;
- (ii) $H^n(\mathcal{F}(k)) = 0$ for $k > -n-1$;
- (iii) $H^i(\mathcal{F}(k)) = 0 \ \forall k, 2 \leq i \leq n-1$, when $n \geq 3$.

Proof. We twist the middle column of the display (3) by $\mathcal{O}_{\mathbb{P}^n}(k)$, then break it into short exact sequences

$$\begin{aligned}
(7) \quad & 0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(k+1-n)^{\oplus a_{1-n}} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(k+2-n)^{\oplus a_{2-n}} \longrightarrow Q_{2-n}(k) \longrightarrow 0 \\
& 0 \longrightarrow Q_{2-n}(k) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(k+3-n)^{\oplus a_{2-n}} \longrightarrow Q_{3-n}(k) \longrightarrow 0 \\
& \quad \quad \quad \vdots \\
& 0 \longrightarrow Q_{-p-1}(k) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(k-p)^{\oplus a_{-p}} \longrightarrow Q_{-p}(k) \longrightarrow 0 \\
& \quad \quad \quad \vdots \\
& 0 \longrightarrow Q_{-2}(k) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(k-1)^{\oplus a_{-1}} \longrightarrow Q_{-1}(k) \longrightarrow 0 \\
& 0 \longrightarrow Q_{-1}(k) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(k)^{\oplus a_0} \longrightarrow Q_0(k) \longrightarrow 0
\end{aligned}$$

where $Q_0 := Q = \text{coker } \alpha_{-1}$.

Step. 1: From the long sequences in cohomology of the first row above, we have

$$H^i(\mathcal{O}_{\mathbb{P}^n}(k+2-n))^{\oplus a_{2-n}} \rightarrow H^i(Q_{2-n}(k)) \rightarrow H^{i+1}(\mathcal{O}_{\mathbb{P}^n}(k+1-n))^{\oplus a_{1-n}} \rightarrow \dots$$

Then, from the vanishing properties of line bundles on \mathbb{P}^n , it follows that

- (i) $H^0(Q_{2-n}(k)) = 0$ for $k < n-2$;
- (ii) $H^n(Q_{2-n}(k)) = 0$ for $k > -1$;
- (iii) $H^i(Q_{2-n}(k)) = 0 \forall k, 1 \leq i \leq n-1$.

Step. 2: Using induction on the remaining rows in (7) it follows that, for $p > 2$,

- (i) $H^0(Q_{p-n}(k)) = 0$ for $k < n-p$;
- (ii) $H^n(Q_{p-n}(k)) = 0$ for $k > -p-1$;
- (iii) $H^i(Q_{p-n}(k)) = 0 \forall k, 1 \leq i \leq n-1$.

Step. 3: From the long exact sequence in cohomology of the lower row in (3) twisted by $\mathcal{O}_{\mathbb{P}^n}(k)$ one has

$$H^{i-1}(\mathcal{O}_{\mathbb{P}^n}(k+1))^{\oplus a_1} \rightarrow H^i(\mathcal{F}(k)) \rightarrow H^i(Q(k)) \rightarrow \dots$$

Using the vanishing obtained in Step. 2 for $Q_0 = Q$, the claims of items (i), (ii), (iii) and (iv) follow.

The last item is obtained by dualizing the lower row of (3). \square

Now let us denote by $\Omega_{\mathbb{P}^n}^{-p}$ the bundle of holomorphic $(-p)$ -forms on \mathbb{P}^n , where $p \leq 0$ in our convention.

Proposition 4.8. *If a coherent sheaf \mathcal{E} on \mathbb{P}^n ($n \geq 2$) satisfies:*

- (i) $H^0(\mathcal{E}(-1)) = H^n(\mathbb{P}^n, \mathcal{E}(-n)) = 0$;
- (ii) $H^q(\mathbb{P}^n, \mathcal{E}(k)) = 0 \quad \forall k, \quad 2 \leq q \leq n-1$ when $n \geq 3$;
- (iii) $H^1(\mathbb{P}^n, \mathcal{E} \otimes \Omega_{\mathbb{P}^n}^{-p}(-p-1)) \neq 0$ for $-n \leq p \leq 0$;

then \mathcal{E} is the cohomology of a perfect extended monad.

Proof. Applying Beilinson's theorem [15, Theorem 3.1.4] to the sheaf $\mathcal{E}(-1)$, one gets a spectral sequence with E_1 -term given by

$$E_1^{p,q} = H^q(\mathcal{E} \otimes \Omega_{\mathbb{P}^n}^{-p}(-p-1)) \otimes \mathcal{O}_{\mathbb{P}^n}(p)$$

which converges to the graded sheaf associated to a filtration of $\mathcal{E}(-1)$ itself.

Twist the Euler sequence for the sheaves of differential forms

$$0 \rightarrow \Omega^p(p) \rightarrow \mathcal{O}_{\mathbb{P}^n}^N \rightarrow \Omega^{p-1}(p) \rightarrow 0 \quad , \quad N = \binom{n+1}{p}$$

by $\mathcal{E}(k-p)$ and use hypotheses (i) and (ii) above to conclude, after long but straightforward calculations with the associated long exact sequences of cohomology, that $E_1^{p,q} = 0$ for $q \neq 1$.

It follows immediately that the Beilinson spectral sequence degenerates already at the E_2 -term, i.e. $E_2 = E_\infty$. Beilinson's theorem then implies that the complex $E_1^{p,1}$ given by

$$(8) \quad V_n \otimes \mathcal{O}_{\mathbb{P}^n}(-n) \rightarrow \cdots \rightarrow V_1 \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \rightarrow V_0 \otimes \mathcal{O}_{\mathbb{P}^n},$$

with $V_p := H^1(\mathcal{E} \otimes \Omega_{\mathbb{P}^n}^{-p}(-p-1))$, $-n \leq p \leq 0$, is exact everywhere except at position $p = -1$, and its cohomology at this position is precisely $\mathcal{E}(-1)$.

The third hypothesis implies that none of the vector spaces V_p vanishes. So twisting the complex (8) by $\mathcal{O}_{\mathbb{P}^n}(1)$, we obtain a perfect extended monad whose cohomology is exactly \mathcal{E} , as desired. \square

5. IDEAL SHEAVES OF ZERO-DIMENSIONAL SUBSCHEMES OF \mathbb{P}^n

We now consider sheaves \mathcal{E} of rank r on \mathbb{P}^n fitting in the following short exact sequence

$$(9) \quad 0 \rightarrow \mathcal{E} \rightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus r} \rightarrow \mathcal{Q} \rightarrow 0,$$

where \mathcal{Q} is a pure torsion sheaf of length c supported on a 0-dimensional subscheme $Z \subset \mathbb{P}^n$.

Note that the Chern character of \mathcal{E} is given by $ch(\mathcal{E}) = r - cH^n$, and that \mathcal{E} is necessarily torsion free. Such sheaves can also be regarded as points in the Quot scheme $Quot^{P=c}(\mathcal{O}_{\mathbb{P}^n}^{\oplus r})$.

In the case $r = 1$, it is clear that \mathcal{E} is the sheaf of ideals in $\mathcal{O}_{\mathbb{P}^n}$ associated to the zero-dimensional subscheme Z , i.e. $\mathcal{Q} = \mathcal{O}_Z$; in this case, we will then denote \mathcal{E} by \mathcal{I}_Z .

Proposition 5.1. *Every sheaf \mathcal{E} on \mathbb{P}^n given by sequence (9) is the cohomology of a perfect extended monad P^\bullet with terms of the form $P^{-i} := V_i \otimes \mathcal{O}_{\mathbb{P}^n}(i)$, $i = 1 - n, \dots, 0, 1$, where*

$$(10) \quad V_i := H^1(\mathcal{E} \otimes \Omega_{\mathbb{P}^n}^{1-i}(-i)) \cong H^0(\mathcal{Q} \otimes \Omega_{\mathbb{P}^n}^{1-i}(-i)).$$

Furthermore, we the following isomorphisms:

$$(11) \quad V_1 \cong H^0(\mathcal{Q})$$

$$(12) \quad V_i \cong \begin{cases} V_1^{\oplus n} \oplus \mathbb{C}^r & \text{for } i = 0 \\ V_1^{\oplus \binom{n}{1-i}} & \text{for } i < 0 \end{cases}$$

In particular, we conclude that

$$\dim V_i = \begin{cases} c & i = 1 \\ nc + r & i = 0 \\ \binom{n}{1-i}c & i < 0 \end{cases}$$

Proof. Conditions (i) and (ii) in Proposition 4.8 follow easily from twisting sequence (9) by $\mathcal{O}_{\mathbb{P}^n}(k)$ and using that fact that \mathcal{Q} is supported in dimension zero. Next, twist sequence (9) by $\Omega_{\mathbb{P}^n}^{-p}(-p-1)$ and use Bott's formula to obtain the isomorphisms in (10).

The isomorphisms (11) and (12) can be proved as follows. First, we have for $i = 1$

$$V_1 := H^1(\mathcal{E}(-1)) \cong H^0(\mathcal{Q}(-i)) \cong H^0(\mathcal{Q})$$

since \mathcal{Q} is supported in dimension zero.

The space V_0 fits in the sequence

$$0 \rightarrow H^0(\mathcal{Q} \otimes \Omega_{\mathbb{P}^n}^1) \rightarrow H^1(\mathcal{E} \otimes \Omega_{\mathbb{P}^n}^1) \rightarrow H^1(\Omega_{\mathbb{P}^n}^1)^{\oplus r} \rightarrow 0$$

obtained from sequence (9) twisted by $\Omega_{\mathbb{P}^n}^1$. On the other hand, we know from the Euler sequence that $H^1(\Omega_{\mathbb{P}^n}^1) \cong H^0(\mathcal{O}_{\mathbb{P}^n})$. Moreover, since

$$H^0(\mathcal{Q} \otimes \Omega_{\mathbb{P}^n}^1) \cong H^0(\mathcal{Q})^{\oplus n} \cong V_1^{\oplus n},$$

it follows that $H^1(\mathcal{E} \otimes \Omega_{\mathbb{P}^n}^{1-i}) \cong V_1^{\oplus n} \oplus \mathbb{C}^r$.

Finally, note that

$$V_{-i} = H^1(\mathcal{E} \otimes \Omega_{\mathbb{P}^n}^{1-i}(-i)) \cong H^0(\mathcal{Q}^{\oplus \binom{n}{-i}}) = V_1^{\oplus \binom{n}{1-i}}.$$

□

In particular, for the case $r = 1$, we have the following Corollary.

Corollary 5.2. *For every zero dimensional subscheme $Z \subset \mathbb{P}^n$, there exists a perfect extended monad P^\bullet of the form*

$$V_{1-n} \otimes \mathcal{O}_{\mathbb{P}^n}(1-n) \xrightarrow{\alpha_{1-n}} V_{2-n} \otimes \mathcal{O}_{\mathbb{P}^n}(2-n) \quad \dots \xrightarrow{\alpha_{-1}} V_0 \otimes \mathcal{O}_{\mathbb{P}^n} \xrightarrow{\alpha_0} V_1 \otimes \mathcal{O}_{\mathbb{P}^n}(1)$$

where $V_1 := H^0(\mathcal{O}_Z)$ and

$$V_{-i} \cong \begin{cases} V_1^{\oplus n} \oplus \mathbb{C} & \text{for } i = 0 \\ V_1^{\oplus \binom{n}{1-i}} & \text{for } i < 0 \end{cases},$$

whose cohomology is the ideal sheaf \mathcal{I}_Z .

6. THE \mathbb{P}^3 CASE

In this section, we fix a hyperplane $\wp \subset \mathbb{P}^n$. We shall describe how to get linear algebraic data out of the perfect extended monad corresponding to a 0-dimensional subscheme $Z \subset \mathbb{P}^3 \setminus \wp$, as in Corollary 5.2.

Let us start by fixing notation; we choose homogeneous coordinates $[z_0; z_1; z_2; z_3]$ on \mathbb{P}^3 in such a way that the hyperplane \wp is given by the equation $z_3 = 0$. We also regard such coordinates as a basis for the space of global sections $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$.

By Corollary 5.2, there is a perfect extended monad P^\bullet with cohomology equal to the ideal sheaf \mathcal{I}_Z . It is given by

$$(13) \quad V_1 \otimes \mathcal{O}_{\mathbb{P}^n}(-2) \xrightarrow{\alpha_{-2}} V_1^{\oplus 3} \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{\alpha_{-1}} (V_1^{\oplus 3} \oplus W) \otimes \mathcal{O}_{\mathbb{P}^n} \xrightarrow{\alpha_0} V_1 \otimes \mathcal{O}_{\mathbb{P}^n}(1)$$

where $\alpha_{-2} \in \text{Hom}(V_1, V_1^{\oplus 3}) \otimes H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$, $\alpha_{-1} \in \text{Hom}(V_1^{\oplus 3}, V_1^{\oplus 3} \oplus W) \otimes H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$ and $\alpha_0 \in \text{Hom}(V_1^{\oplus 3} \oplus W, V_1) \otimes H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$. Then we can write the α 's as:

$$\begin{aligned}\alpha_{-2} &= \alpha_{-2}^0 z_0 + \alpha_{-2}^1 z_1 + \alpha_{-2}^2 z_2 + \alpha_{-2}^3 z_3; \\ \alpha_{-1} &= \alpha_{-1}^0 z_0 + \alpha_{-1}^1 z_1 + \alpha_{-1}^2 z_2 + \alpha_{-1}^3 z_3; \\ \alpha_0 &= \alpha_0^0 z_0 + \alpha_0^1 z_1 + \alpha_0^2 z_2 + \alpha_0^3 z_3,\end{aligned}$$

The conditions $\alpha_{-1} \circ \alpha_{-2} = 0$ and $\alpha_0 \circ \alpha_{-1} = 0$, which guarantee that (13) is a complex, are equivalent to

$$(14) \quad \alpha_{1-i}^k \circ \alpha_{-i}^k = 0 \quad \forall k, i \quad \text{and} \quad \alpha_{1-i}^k \circ \alpha_{-i}^l + \alpha_{1-i}^l \circ \alpha_{-i}^k = 0 \quad \forall i, k \neq l.$$

We also have to impose the condition $\ker \alpha_{-1} = \text{Im } \alpha_{-2}$, since $\mathcal{H}^{-1}(P^\bullet) = 0$.

Restricting P^\bullet to the plane $\wp \simeq \mathbb{P}^2$ we get the following 1-extended monad on \wp :

$$(15) \quad V_1 \otimes \mathcal{O}_\wp(-2) \xrightarrow{\alpha_{-2}|_\wp} V_1^{\oplus 3} \otimes \mathcal{O}_\wp(-1) \xrightarrow{\alpha_{-1}|_\wp} (V_1^{\oplus 3} \oplus W) \otimes \mathcal{O}_\wp \xrightarrow{\alpha_0|_\wp} V_1 \otimes \mathcal{O}_\wp(1)$$

and the maps of this complex are just given by

$$\begin{aligned}\alpha_{-2}|_\wp &= \alpha_{-2}^0 z_0 + \alpha_{-2}^1 z_1 + \alpha_{-2}^2 z_2; \\ \alpha_{-1}|_\wp &= \alpha_{-1}^0 z_0 + \alpha_{-1}^1 z_1 + \alpha_{-1}^2 z_2; \\ \alpha_0|_\wp &= \alpha_0^0 z_0 + \alpha_0^1 z_1 + \alpha_0^2 z_2.\end{aligned}$$

The resolution and the monad associated to the perfect extended monad P^\bullet are given by, respectively,

$$(16) \quad 0 \longrightarrow V_1 \otimes \mathcal{O}_\wp(-2) \xrightarrow{\alpha_{-2}|_\wp} V_1^{\oplus 3} \otimes \mathcal{O}_\wp(-1) \xrightarrow{J_{-1}|_\wp} \mathcal{G}|_\wp \longrightarrow 0$$

$$(17) \quad \mathcal{G}|_\wp \xrightarrow{I_{-1}|_\wp} (V_1^{\oplus 3} \oplus W) \otimes \mathcal{O}_\wp \xrightarrow{\alpha_0|_\wp} V_1 \otimes \mathcal{O}_\wp(1) \quad .$$

Lemma 6.1. *The sheaf $\mathcal{G}|_\wp$ is locally free and satisfies*

- (i) $H^0(\wp, \mathcal{G}|_\wp) = H^1(\wp, \mathcal{G}|_\wp) = H^2(\wp, \mathcal{G}|_\wp) = 0$;
- (ii) $H^1(\wp, \mathcal{G}|_\wp^*) = H^2(\wp, \mathcal{G}|_\wp^*) = 0$, and $h^0(\wp, \mathcal{G}|_\wp^*) = 3c$.

Proof. Taking the restriction of the display of the perfect monad to the plane \wp one has $\mathcal{I}|_\wp = \mathcal{O}|_\wp$, since $\text{supp}(Z) \cap \wp = \emptyset$. Moreover, from the lowest row of the restricted display, namely

$$0 \rightarrow \mathcal{O}|_\wp \rightarrow Q|_\wp \rightarrow V_1 \otimes \mathcal{O}|_\wp \rightarrow 0,$$

it follows that $Q|_\wp$ is a locally free sheaf. Furthermore, from the middle column of the restricted display, namely

$$0 \rightarrow \mathcal{G}|_\wp \rightarrow (V_1^{\oplus 3} \oplus W) \otimes \mathcal{O}|_\wp \rightarrow Q|_\wp \rightarrow 0,$$

it also follows the sheaf $\mathcal{G}|_\wp$ is locally free.

The first item follows from the long exact sequence in cohomology of the associated resolution (16) and the fact that $H^i(\wp, \mathcal{O}|_\wp(k)) = 0$, for $i = 0, 1, 2$ and $k = -1, -2$.

For the second item, dualize the exact sequence (16) and apply the global sections functor Γ to obtain the exact sequence

$$(18) \quad 0 \rightarrow H^0(\wp, \mathcal{G}|_\wp^*) \rightarrow (V_1^*)^{\oplus 3} \otimes H^0(\wp, \mathcal{O}|_\wp(1)) \rightarrow V_1^* \otimes H^0(\wp, \mathcal{O}|_\wp(2)) \rightarrow H^1(\wp, \mathcal{G}|_\wp^*) \rightarrow 0.$$

and $H^2(\wp, \mathcal{G}|_{\wp}^*) = 0$, since $H^{1,2}(\wp, \mathcal{O}|_{\wp}(1)) = H^{1,2}(\wp, \mathcal{O}|_{\wp}(1)) = 0$. On the other hand, from the dual display of the associated monad (17) one has the exact sequence

$$(19) \quad 0 \rightarrow Q|_{\wp}^* \rightarrow (V_1^* \oplus W^*) \otimes \mathcal{O}|_{\wp} \rightarrow \mathcal{G}|_{\wp}^* \rightarrow 0$$

where $Q := \text{coker } \alpha_{-1}$. Moreover $Q|_{\wp}^* = V_1^* \otimes \mathcal{O}|_{\wp}(1) \oplus \mathcal{O}|_{\wp}$ since $Q|_{\wp} \in \text{Ext}^1(V_1 \otimes \mathcal{O}|_{\wp}(1), \mathcal{O}|_{\wp}) = V_1^* \otimes H^1(\wp, \mathcal{O}|_{\wp}(-1)) = 0$. Then, from the long exact sequence in cohomology associated to (19), it follows that $H^0(\wp, \mathcal{G}|_{\wp}^*)$ fits in the exact sequence

$$(20) \quad 0 \rightarrow \mathbb{C} \rightarrow (V_1^*)^{\oplus 3} \oplus W \rightarrow H^0(\wp, \mathcal{G}|_{\wp}^*) \rightarrow 0,$$

hence $h^0(\wp, \mathcal{G}|_{\wp}^*) = 3c$ and from (18) it follows that $h^1(\wp, \mathcal{G}|_{\wp}^*) = 0$. \square

Remark that the sequence (18) becomes just

$$(21) \quad 0 \rightarrow H^0(\wp, \mathcal{G}|_{\wp}^*) \xrightarrow{i} (V_1^*)^{\oplus 3} \oplus (V_1^*)^{\oplus 3} \oplus (V_1^*)^{\oplus 3} \xrightarrow{j} (V_1^*)^{\oplus 3} \oplus (V_1^*)^{\oplus 3} \rightarrow 0,$$

since $H^0(\wp, \mathcal{O}|_{\wp}(1)) \simeq \mathbb{C}^3$, and $H^0(\wp, \mathcal{O}|_{\wp}(2)) \simeq \mathbb{C}^6$. So one can identify $H^0(\wp, \mathcal{G}|_{\wp}^*)$ with $(V_1^*)^{\oplus 3}$. Furthermore, by (20), one can identify $W = H^0(\wp, \mathcal{I}_Z|_{\wp}) \cong \mathbb{C}$, since $Z \cap \wp = \emptyset$.

Combining sequences (21) and (20), and dualizing the resulting sequence one gets

$$(22) \quad 0 \rightarrow V_1^{\oplus 3} \oplus V_1^{\oplus 3} \xrightarrow{i} V_1^{\oplus 3} \oplus V_1^{\oplus 3} \oplus V_1^{\oplus 3} \xrightarrow{j} V_1^{\oplus 3} \oplus W \rightarrow \mathbb{C} \rightarrow 0,$$

The maps i and j are just $H^0(\alpha_{-2})$ and $H^0(\alpha_{-1})$, respectively. Thus we have

$$\ker H^0(\alpha_{-2}) = \ker \alpha_{-2}^0 \cap \ker \alpha_{-2}^1 \cap \ker \alpha_{-2}^2 = \{0\},$$

and

$$\ker H^0({}^t\alpha_{-1}) = \ker {}^t\alpha_{-1}^0 \cap \ker {}^t\alpha_{-1}^1 \cap \ker {}^t\alpha_{-1}^2 = \mathbb{C}.$$

The subscript t , in the last equation, stands for transposition. Remark also that the sequence (22) reflects the fact the complex (15) is exact at degree -1 , i.e., $\alpha_{-1} \circ \alpha_{-2} = 0$.

We can then choose the maps α_{-1}^j in the following way. First,

$$\alpha_{-2}^0, \quad \alpha_{-2}^1, \quad \alpha_{-2}^2 : V_1 \rightarrow V_1 \oplus V_1 \oplus V_1,$$

with:

$$(23) \quad \alpha_{-2}^0 = \begin{pmatrix} 0 \\ 0 \\ \mathbb{I}_{V_1} \end{pmatrix} \quad \alpha_{-2}^1 = \begin{pmatrix} 0 \\ -\mathbb{I}_{V_1} \\ 0 \end{pmatrix} \quad \alpha_{-2}^2 = \begin{pmatrix} \mathbb{I}_{V_1} \\ 0 \\ 0 \end{pmatrix},$$

and where \mathbb{I}_{V_1} denotes the identity in $\text{End}(V_1)$.

One also has

$$\alpha_{-1}^0, \quad \alpha_{-1}^1, \quad \alpha_{-1}^2 : V_1 \oplus V_1 \oplus V_1 \rightarrow V_1 \oplus V_1 \oplus V_1 \oplus \mathbb{C}$$

given by

$$(24) \quad \begin{aligned} \alpha_{-1}^0 &= \begin{pmatrix} 0 & 0 & 0 \\ \mathbb{I}_{V_1} & 0 & 0 \\ 0 & \mathbb{I}_{V_1} & 0 \\ 0 & 0 & 0 \end{pmatrix} & \alpha_{-1}^1 &= \begin{pmatrix} -\mathbb{I}_{V_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathbb{I}_{V_1} \\ 0 & 0 & 0 \end{pmatrix} \\ \alpha_{-1}^2 &= \begin{pmatrix} 0 & -\mathbb{I}_{V_1} & 0 \\ 0 & 0 & -\mathbb{I}_{V_1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Finally, for

$$\alpha_0^0, \quad \alpha_0^1, \quad \alpha_0^2 : V_1 \oplus V_1 \oplus V_1 \oplus \mathbb{C} \rightarrow V_1$$

one has

$$(25) \quad \begin{aligned} \alpha_0^0 &= \begin{pmatrix} -\mathbb{I}_{V_1} & 0 & 0 & 0 \end{pmatrix} & \alpha_0^1 &= \begin{pmatrix} 0 & -\mathbb{I}_{V_1} & 0 & 0 \end{pmatrix} \\ \alpha_0^2 &= \begin{pmatrix} 0 & 0 & -\mathbb{I}_{V_1} & 0 \end{pmatrix}. \end{aligned}$$

Now, to complete our construction, we have to add the maps α_{-2}^3 , α_{-1}^3 and α_0^3 such that conditions (14) are satisfied. By putting

$$(26) \quad \alpha_{-2}^3 = \begin{pmatrix} -B_2 \\ B_1 \\ -B_0 \end{pmatrix}; \quad \alpha_{-1}^3 = \begin{pmatrix} B_1 & B_2 & 0 \\ -B_0 & 0 & -B_0 \\ 0 & B_2 & -B_1 \\ 0 & 0 & 0 \end{pmatrix}; \quad \alpha_0^3 = \begin{pmatrix} B_0 & B_1 & B_2 & I \end{pmatrix},$$

where $B_i \in \text{End}(V_1)$ and $I \in \text{Hom}(\mathbb{C}, V_1)$, then all the equations are satisfied, since $\alpha_{-1}^3 \circ \alpha_{-2}^3 = 0$ and $\alpha_0^3 \circ \alpha_{-1}^3 = 0$ are equivalent to

$$(27) \quad [B_0, B_1] = 0; \quad [B_0, B_2] = 0; \quad [B_1, B_2] = 0,$$

Summing up what we have done so far, for a given 0-dimensional subscheme $Z \subset \mathbb{P}^3 \setminus \wp$ we have constructed a perfect extended monad P^\bullet of the form

$$(28) \quad V_1 \otimes \mathcal{O}_{\mathbb{P}^n}(-2) \xrightarrow{\alpha_{-2}} V_1^{\oplus 3} \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{\alpha_{-1}} (V_1^{\oplus 3} \oplus W) \otimes \mathcal{O}_{\mathbb{P}^n}^{\alpha_0} \rightarrow V_1 \otimes \mathcal{O}_{\mathbb{P}^n}(1)$$

where the maps α_{-2} , α_{-1} and α_0 are given by ¹:

$$(29) \quad \alpha_{-2} = \begin{pmatrix} -B_2 z_3 + z_2 \\ B_1 z_3 - z_1 \\ -B_0 z_3 + z_0 \end{pmatrix}; \quad \alpha_{-1} = \begin{pmatrix} B_1 z_3 - z_1 & B_2 z_3 - z_2 & 0 \\ -B_0 z_3 + z_0 & 0 & B_2 z_3 - z_2 \\ 0 & -B_0 z_3 + z_0 & -B_1 z_3 + z_1 \\ 0 & 0 & 0 \end{pmatrix};$$

$$\alpha_0 = \begin{pmatrix} B_0 z_3 - z_0 & B_1 z_3 - z_1 & B_2 z_3 - z_2 & I z_3 \end{pmatrix}.$$

It only remains for us to show the the ADHM datum

$$(B_0, B_1, B_2, I) \in \text{End}(V_1)^{\oplus 3} \oplus \text{Hom}(\mathbb{C}, V_1)$$

obtained from the above construction is indeed stable. Such claim will follow from the following observation.

¹ We omit writing the identity in front of the coordinates so $z_i \mathbb{I}_{V_1}$ will just be written z_i

Lemma 6.2. *The map α_0 given above is surjective if and only if the ADHM datum (B_0, B_1, B_2, I) is stable.*

Proof. Recall that a map of sheaves is surjective if and only if it is surjective at every fiber.

So if α_0 is not surjective, then there is a point $z = [z_0; z_1; z_2; z_3] \in \mathbb{P}^3$ such that $\alpha_0(z)$ is not surjective, while its dual map α_0^* is not injective. Hence there exists a vector $\bar{v} \in V^*$ such that $(B_i^* z_3 - z_i)\bar{v} = 0$, where $i = 0, 1, 2$, and $I^*\bar{v} = 0$. Then the subspace $\bar{S} \subsetneq V^*$ generated by all such vectors is B_i^* -invariant, for $i = 0, 1, 2$, while the restriction $I^*|_{\bar{S}}$ of I^* to \bar{S} is zero.

Now, consider the following subspace of V :

$$S = \{v \in V \mid \bar{v}(v) = 0, \forall \bar{v} \in \bar{S}\}.$$

It follows that S is B_i -invariant, for $i = 0, 1, 2$, since $B_i^*\bar{v}(v) = \bar{v}(B_i v) = 0$ for $\bar{v} \in \bar{S}$ and $v \in S$. Moreover $I(1) \in S$ since $I^*|_{\bar{S}} = 0$. Thus (B_0, B_1, B_2, I) is not stable.

Conversely, suppose that (B_0, B_1, B_2, I) is not stable. Then there exists a B_i -invariant subspace $S \subsetneq V$, for $i = 0, 1, 2$, such that $\text{Im } I \subseteq S$. Set

$$\bar{S} = \{\bar{v} \in V^* \mid \bar{v}(v) = 0, \forall v \in S\}.$$

Then \bar{S} is B_i^* -invariant and $\bar{S} \subset \ker I^*$. Since the B_i 's commutes, there exists a vector $\bar{v} \in V^*$ such that $B_i^*\bar{v} = \lambda_i \bar{v}$ for some $\lambda_i \in \mathbb{C}$, for $i = 0, 1, 2$. Hence the map $\alpha_0^*(\lambda_0; \lambda_1; \lambda_2; 1)$ is not injective, and equivalently, α_0 is not surjective. \square

Theorem 6.3 (Inverse construction). *To a stable ADHM datum $X = (B_0, B_1, B_2, I) \in \mathcal{V}(3, c)^{st}$ one can associate the perfect extended monad (28) with maps $\alpha_{-2}, \alpha_{-1}, \alpha_0$ given as in (29), such that its cohomology is an ideal sheaf whose restriction to $\mathbb{C}^3 = \mathbb{P}^3 \setminus \emptyset$ is isomorphic to the one given by Theorem 3.1.*

Proof. Given the stable ADHM datum (B_0, B_1, B_2, I) , one can put together the maps α_{-2}, α_{-1} and α_0 and a perfect extended monad like (28). Restricting the obtained perfect monad to \mathbb{C}^3 one has the complex

$$(30) \quad 0 \longrightarrow \mathbb{C}^c \otimes \mathcal{O}_{\mathbb{C}^3} \xrightarrow{a_{-2}} \begin{array}{c} \mathbb{C}^c \otimes \mathcal{O}_{\mathbb{C}^3} \\ \oplus \\ \mathbb{C}^c \otimes \mathcal{O}_{\mathbb{C}^3} \\ \oplus \\ \mathbb{C}^c \otimes \mathcal{O}_{\mathbb{C}^3} \\ \oplus \\ \mathbb{C}^c \otimes \mathcal{O}_{\mathbb{C}^3} \end{array} \xrightarrow{a_{-1}} \begin{array}{c} \mathbb{C}^c \otimes \mathcal{O}_{\mathbb{C}^3} \\ \oplus \\ \mathbb{C}^c \otimes \mathcal{O}_{\mathbb{C}^3} \\ \oplus \\ \mathbb{C}^c \otimes \mathcal{O}_{\mathbb{C}^3} \\ \oplus \\ \mathbb{C}^c \otimes \mathcal{O}_{\mathbb{C}^3} \end{array} \xrightarrow{a_0} \mathbb{C}^c \otimes \mathcal{O}_{\mathbb{C}^3} \longrightarrow 0.$$

Moreover, by projecting on the fourth summand in the degree 0 term, one has the injection $\ker a_0 / \text{Im } a_{-1} \hookrightarrow \mathcal{O}_{\mathbb{C}^3}$. We denote its image by \mathcal{J} . Then, it is clear that \mathcal{J} is an ideal of c points in \mathbb{C}^3 .

To prove that \mathcal{J} is isomorphic to $\ker \Phi_X = \{f \in \mathcal{O}_{\mathbb{C}^3} \mid f(B_0, B_1, B_2) = 0\}$, let us suppose, first, that $f \in \mathcal{J}$. Then, there exists three vectors $u_0(z), u_1(z), u_2(z) \in V \otimes \mathcal{O}_{\mathbb{C}^3}$ such that $f(z)I(1) = (B_0 - z_0)u_0(z) + (B_1 - z_1)u_1(z) + (B_2 - z_2)u_2(z)$, since f represents an element in $\ker a_0$. Hence $f(B_0, B_1, B_2)I(1) = 0$. But (B_0, B_1, B_2, I) is stable and $B_0^{l_0} B_1^{l_1} B_2^{l_2}$ span all V , for $l_0, l_1, l_2 \geq 0$. Thus $f(B_0, B_1, B_2) = 0$, i.e., $f \in \ker \Phi_X$.

On the other hand, let $f \in \mathcal{O}_{\mathbb{C}^3}$ such that $f(B_0, B_1, B_2)$. One has (unless otherwise specified, all sums are taken with respect to the indices $l_0, l_1, l_2 \geq 0$):

$$\begin{aligned} f(z_0, z_1, z_2)\mathbb{I}_V &= \sum a_{l_0 l_1 l_2} z_0^{l_0} z_1^{l_1} z_2^{l_2} \mathbb{I}_V \\ &= \sum a_{l_0 l_1 l_2} (z_0 - B_0 + B_0)^{l_0} (z_1 - B_1 + B_1)^{l_1} (z_2 - B_2 + B_2)^{l_2} \\ &= \sum a_{l_0 l_1 l_2} \left\{ \sum_{i=0}^{l_0} \alpha_i (z_0 - B_0)^i B_0^{l_0-i} \right\} \cdot \left\{ \sum_{j=0}^{l_1} \beta_j (z_1 - B_1)^j B_1^{l_1-j} \right\} \\ &\quad \cdot \left\{ \sum_{k=0}^{l_2} \gamma_k (z_2 - B_2)^k B_2^{l_2-k} \right\} \end{aligned}$$

where we used the expansion $(a+b)^n = \sum_{i=0}^n \alpha_i a^i b^{n-i}$, $\alpha_i = \binom{n}{i}$ in the third line. Expanding again

$$\left\{ \sum_{i=0}^{l_q} \alpha_i (z_q - B_q)^i B_q^{l_q-i} \right\} = B_q^{l_q} + (z_q - B_q)^{l_q} + \left\{ \sum_{i=1}^{l_q-1} \alpha_i (z_q - B_q)^i B_q^{l_q-i} \right\}$$

with $q = 0, 1, 2$, thus one obtains

$$\begin{aligned} f(z_0, z_1, z_2)\mathbb{I}_V &= \sum a_{l_0 l_1 l_2} B_0^{l_0} B_1^{l_1} B_2^{l_2} + \sum_{q=0}^2 (z_q - B_q) A_q(z) \\ &= f(B_0, B_1, B_2) + \sum_{q=0}^2 (z_q - B_q) A_q(z) \end{aligned}$$

for some vectors $A_q \in \text{End}(V) \otimes \mathcal{O}_{\mathbb{C}^3}$, $q = 0, 1, 2$. But $f(B_0, B_1, B_2) = 0$ by hypothesis. If we put $u_q = A_q I(1)$, $q = 0, 1, 2$, then

$$f(z_0, z_1, z_2)I(1) = (z_0 - B_0)u_0(z) + (z_1 - B_1)u_1(z) + (z_2 - B_2)u_2(z).$$

Hence, $f \in \mathcal{J}$. □

The automorphisms of P^\bullet are clearly given by the action of the group $GL(V_1)$. Since, by Corollary 4.6, the cohomology functor is fully faithful, we recover the correspondence, given in Section 2, between equivalence classes of ideal sheaves \mathcal{I}_Z and the space $\mathcal{M}(3, c)$ defined as the quotient $\mathcal{V}(3, c)^{st}/GL(V_1)$, in the 3-dimensional case.

We complete this section by writing down the maps α_0 and α_{-1} in the more general n -dimensional case. Starting with a hyperplane $\wp \subset \mathbb{P}^n$ and a 0-dimensional subscheme $Z \subset \mathbb{P}^n \setminus \wp$, the maps α_{-i} in the corresponding perfect extended monad can also be constructed as done above for the 3-dimensional case:

$$(31) \quad \begin{aligned} \alpha_0 &= \begin{pmatrix} B_0 z_n - z_0 & B_1 z_n - z_1 & \cdots & B_{n-1} z_n - z_{n-1} & I z_n \end{pmatrix} \cdot \\ \alpha_{-1} &= \begin{pmatrix} A_0 & A_1 & \cdots & A_{n-2} \\ 0 & 0 & \cdots & 0 \end{pmatrix}. \end{aligned}$$

where each block A_i , $0 \leq i \leq n-2$ is an $[(n-i) \cdot c \times n \cdot c]$ -matrix of the form

$$A_i = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ B_{i+1}z_n - z_{i+1} & B_{i+2}z_n - z_{i+2} & B_{i+3}z_n - z_{i+3} & \cdots & B_{n-1}z_n - z_{n-1} \\ -B_i z_n + z_i & 0 & 0 & \cdots & 0 \\ 0 & -B_i z_n + z_i & 0 & \cdots & 0 \\ 0 & 0 & -B_i z_n + z_i & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & -B_i z_n + z_i \end{pmatrix}$$

One can similarly show that $\alpha_0 \circ \alpha_{-1} = 0 \Leftrightarrow [B_i, B_j] = 0$, for all $0 \leq i, j \leq n-1$. and that the map α_0 is surjective if and only if the ADHM datum (B_0, \dots, B_{n-1}, I) is stable.

Once again, this reflects the set theoretic bijection between the Hilbert scheme of length c zero-dimensional subschemes of $\mathbb{C}^n \simeq \mathbb{P}^n \setminus \wp$ and the quotient space $\mathcal{M}(n, c) := \mathcal{V}(n, c)^{st} / GL(V_1)$.

7. REPRESENTABILITY OF THE HILBERT FUNCTOR OF POINTS

Let us start this Section by introducing notation; for every two sheaves, \mathcal{F} on \mathbb{P}^n and \mathcal{G} on a scheme S , we put $\mathcal{F} \boxtimes \mathcal{G} := p^* \mathcal{F} \otimes q^* \mathcal{G}$, where $p : \mathbb{P}^n \times S \rightarrow \mathbb{P}^n$ is the projection on the first factor and q is the projection $\mathbb{P}^n \times S \rightarrow S$ on the second one. We also denote by $k(s)$ the residue field of a closed point $s \in S$.

Using the ingredients developed in the previous sections, we now proceed to prove that $\mathcal{M}(n, c)$ represents the Hilbert functor

$$\text{Hilb}_{\mathbb{C}^n}^{[c]} : \mathfrak{Sch} \rightarrow \mathfrak{Set}$$

from the category of schemes \mathfrak{Sch} to the category of sets \mathfrak{Set} , which associates to every scheme S the set

$$\text{Hilb}_{\mathbb{C}^n}^{[c]}(S) := \left\{ Z \subset \mathbb{C}^n \times S \left| \begin{array}{l} Z \text{ is a closed subscheme,} \\ Z \hookrightarrow \mathbb{C}^n \times S \\ \begin{array}{ccc} \pi \downarrow & & \downarrow q \\ S & \simeq & S \end{array} \text{ with } \pi \text{ flat, and} \\ \chi(\mathcal{O}_{\pi^{-1}(s)} \otimes \mathcal{O}_{\mathbb{C}^n}(m)) = c, \quad \forall m \in \mathbb{Z}. \end{array} \right. \right\}$$

of flat families of 0-dimensional subschemes of \mathbb{C}^n .

For any noetherian scheme S of finite type over the field of complex numbers \mathbb{C} , consider the following diagram:

$$\begin{array}{ccc} \mathbb{P}^n \times \mathbb{P}^n \times S & \xrightarrow{pr_{13}} & \mathbb{P}^n \times S \\ pr_{23} \downarrow & & \downarrow q \\ \mathbb{P}^n \times S & \xrightarrow{q} & S \end{array}$$

and the relative Euler sequence:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n \times S}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n \times S}^{\oplus(n+1)} \longrightarrow T\mathbb{P}^n(-1) \boxtimes \mathcal{O}_S \longrightarrow 0$$

where $T\mathbb{P}^n(-1)$ is tangent bundle. One has the following:

Theorem 7.1 (Relative Beilinson's Theorem.). *For every coherent sheaf \mathcal{F} on $\mathbb{P}^n \times S$ there is a spectral sequence $E_r^{i,j}$ with E_1 -term*

$$E_1^{i,j} = \mathcal{O}_{\mathbb{P}^n}(i) \boxtimes \mathcal{R}^j q_*(\mathcal{F} \otimes \Omega_{\mathbb{P}^n \times S/S}^{-i}(-i))$$

which converges to

$$E_\infty^{i,j} = \begin{cases} \mathcal{F} & i+j=0 \\ 0 & \text{otherwise.} \end{cases}$$

Let \mathcal{J} be an S -flat family of ideal sheaves of 0-dimensional subschemes of \mathbb{P}^n of length c , for a noetherian scheme S of finite type.

Theorem 7.2. *There exists an 1-extended monad given by*

$$\begin{aligned} & \mathcal{O}_{\mathbb{P}^n}(1-n) \boxtimes \mathcal{R}^1 q_*(\mathcal{J} \otimes \Omega_{\mathbb{P}^n \times S/S}^n(n-1)) \rightarrow \mathcal{O}_{\mathbb{P}^n}(2-n) \boxtimes \mathcal{R}^1 q_*(\mathcal{J} \otimes \Omega_{\mathbb{P}^n \times S/S}^{n-1}(n-2)) \rightarrow \cdots \\ (32) \quad & \cdots \rightarrow \mathcal{O}_{\mathbb{P}^n} \boxtimes \mathcal{R}^1 q_*(\mathcal{J} \otimes \Omega_{\mathbb{P}^n \times S/S}^1) \rightarrow \mathcal{O}_{\mathbb{P}^n}(1) \boxtimes \mathcal{R}^1 q_{*1}(\mathcal{J} \otimes p^* \mathcal{O}_{\mathbb{P}^n}(-1)) \end{aligned}$$

such that it's cohomology is exactly the family \mathcal{J} .

Proof. By the relative Beilinson theorem, we only need the S -flatness of \mathcal{J} and the fact that at point $s \in S$ one has

$$\mathcal{R}^1 q_*(\mathcal{J} \otimes \Omega_{\mathbb{P}^n \times S/S}^{-i}(1-i)) \otimes k(s) \simeq H^1(\mathbb{P}^n, \mathcal{I}_{Z(s)} \otimes \Omega_{\mathbb{P}^n}^{-i}(1-i)),$$

where $Z(s)$ is the 0-dimensional subscheme of \mathbb{P}^n corresponding to the point $s \in S$. The rest of the proof follows from the vanishing properties of Lemma 5.1. \square

Therefore, on every point $s \in S$, one has a perfect extended monad $P^\bullet(s)$ given by

$$\begin{aligned} & H^1(\mathcal{I}_{Z(s)} \otimes \Omega_{\mathbb{P}^n}^n(n-1)) \otimes \mathcal{O}_{\mathbb{P}^n}(1-n) \rightarrow H^1(\mathcal{I}_{Z(s)} \otimes \Omega_{\mathbb{P}^n}^{n-1}(n-2)) \otimes \mathcal{O}_{\mathbb{P}^n}(2-n) \rightarrow \cdots \\ & \cdots \rightarrow H^1(\mathcal{I}_{Z(s)} \otimes \Omega_{\mathbb{P}^n}^1) \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \rightarrow H^1(\mathcal{I}_{Z(s)} \otimes \mathcal{O}_{\mathbb{P}^n}(-1)) \otimes \mathcal{O}_{\mathbb{P}^n}(1) \end{aligned}$$

Moreover, in the case of the space $\mathcal{V}(n, c)^{st}$ defined by one have the *universal extended monad*

$$\begin{aligned} & \mathcal{O}_{\mathbb{P}^n}(1-n) \boxtimes (V_1 \otimes \mathcal{O}_{\mathcal{V}(n, c)^{st}}) \rightarrow \mathcal{O}_{\mathbb{P}^n}(2-n) \boxtimes (V_1^{\oplus \binom{n}{n-1}} \otimes \mathcal{O}_{\mathcal{V}(n, c)^{st}}) \rightarrow \cdots \\ (33) \quad & \cdots \rightarrow \mathcal{O}_{\mathbb{P}^n} \boxtimes ((V_1^{\oplus n} \oplus W) \otimes \mathcal{O}_{\mathcal{V}(n, c)^{st}}) \rightarrow \mathcal{O}_{\mathbb{P}^n}(1) \boxtimes (V_1 \otimes \mathcal{O}_{\mathcal{V}(n, c)^{st}}) \end{aligned}$$

Finally we have the following

Theorem 7.3. *The scheme $\mathcal{M}(n, c)$ is a fine moduli space for the Hilbert functor $\text{Hilb}_{\mathbb{C}^n}^{[c]}$ of c points on \mathbb{C}^n .*

Proof. The proof is similar, *mutatis mutandis*, to that of [7, Theorem 4.2]. \square

It follows by universality of the Hilbert scheme that

Corollary 7.4. $\text{Hilb}^{[c]}(\mathbb{C}^n) \simeq \mathcal{M}(n, c)$ as schemes.

8. THE HILBERT–CHOW MAP

Let \mathcal{S}^c denote the group of permutations on c elements, and consider the symmetric product of c copies of \mathbb{C}^n :

$$S^{(c)}\mathbb{C}^n := (\mathbb{C}^n \times \mathbb{C}^n \times \dots \times \mathbb{C}^n) / \mathcal{S}^c.$$

In this section, we show how one can describe the Hilbert–Chow morphism

$$HC : \text{Hilb}^{[c]}(\mathbb{C}^n) \rightarrow S^{(c)}\mathbb{C}^n$$

in terms of the linear data $[(B_0, B_1, \dots, B_{n-1}, I)]$.

Recall that a partition $\nu = (\nu_1, \nu_2, \dots, \nu_k)$, with $\nu_1 \geq \nu_2 \geq \dots \geq \nu_k \geq 0$, of c of length k , gives a stratum

$$Z_\nu^{(c)} = \left\{ \sum_{i=1}^k \nu_i [p_i] \in S^{(n)}\mathbb{C}^n \mid p_i \neq p_j \text{ for } j \neq i \right\}$$

of the symmetric product $S^{(c)}\mathbb{C}^n$ corresponding to k ordered points p_1, p_2, \dots, p_k , in \mathbb{C}^n with multiplicities $\nu_1, \nu_2, \dots, \nu_k$, respectively. There is a set theoretic stratification

$$\text{Hilb}^{[c]}(\mathbb{C}^n) = \bigsqcup_{\nu} U_{\nu}^{[c]},$$

where each stratum $U_{\nu}^{[c]}$ is given by the inverse image $HC^{-1}(Z_{\nu}^{(c)})$.

Now let $[(B_0, B_1, \dots, B_{n-1}, I)]$ be a datum class in $\text{Hilb}^{[c]}(\mathbb{C}^n)$; the endomorphisms B_i , $i = \{0, 1, \dots, n-1\}$ can be put simultaneously into Jordan forms, since the B_i 's commutes two by two.

Suppose there are k eigenvalues $\lambda_1^i, \lambda_2^i, \dots, \lambda_k^i$ of each endomorphism B_i , that is, for a fixed $i \in \{0, 1, \dots, n-1\}$, each λ_l^i , $l \in \{1, 2, \dots, k\}$ corresponds to a Jordan block. Then, to every Jordan block one can associate its dimension ν_l , which does not depend on i by the commuting property of the B_i 's.

The Hilbert–Chow map can be represented by:

$$HC : \quad \text{Hilb}_{\mathbb{C}^n}^{[c]} \longrightarrow S^{(c)}\mathbb{C}^n$$

$$[(B_0, B_1, \dots, B_{n-1}, I)] \longmapsto \sum_l^k \nu_l [p_l].$$

where $\{p_l = (\lambda_1^0, \lambda_2^0, \dots, \lambda_l^{n-1})\}_{l=1, \dots, k}$ is a set of point in \mathbb{C}^n , and indeed, $\sum_l^k \nu_l [p_l]$ is the topological support of the zero dimensional subscheme corresponding to the datum $[Z]$.

Of course, when all the eigenvalues are distinct, the endomorphisms B_i are all, simultaneously diagonalizable and the multiplicity $\nu_l = 1$, for all $l = 1, \dots, c$. Hence $\sum_l^c [p_l]$ is a point in the smooth stratum $S^{(c)}\mathbb{C}^n \setminus \Delta$, where $\Delta \subset S^{(c)}\mathbb{C}^n$ is the big diagonal.

9. IRREDUCIBLE COMPONENTS OF THE HILBERT SCHEME OF POINTS

The variety $\mathcal{C}(n, c)$ of n commuting $c \times c$ matrices have been much studied by various authors since a 1961 paper by Gerstenhaber [5]. The results concerning the irreducibility of $\mathcal{C}(n, c)$ can be summarized as follows:

- $\mathcal{C}(2, c)$ is irreducible for every c (originally proved by Motzkin and Taussky [11], see also [5]);

- $\mathcal{C}(3, c)$ is irreducible for $c \leq 10$ and reducible for $c \geq 30$, see [8, 16] and the references therein;
- for $n \geq 4$, $\mathcal{C}(n, c)$ is irreducible if and only if $c \leq 3$ [5].

In particular, determining the highest possible value of c for which $\mathcal{C}(3, c)$ is irreducible is an important open problem.

On the other hand, much less is known about the irreducibility of the Hilbert scheme $\text{Hilb}^{[c]}(\mathbb{C}^n)$ of c points on \mathbb{C}^n , see for instance [1, Section 7] and the references therein. It is well known that $\text{Hilb}^{[c]}(\mathbb{C}^2)$ is irreducible for every c [4] and that, for $n \geq 3$, $\text{Hilb}^{[c]}(\mathbb{C}^n)$ is reducible for c sufficiently large [9]; in particular, Iarrobino has shown that $\text{Hilb}^{[78]}(\mathbb{C}^3)$ is reducible [10]. Similarly, determining the highest possible value of c for which $\text{Hilb}^{[c]}(\mathbb{C}^3)$ is irreducible is also an important open question.

In this section, we connect the two problems through the following result.

Proposition 9.1. *The number of irreducible components of $\text{Hilb}^{[c]}(\mathbb{C}^n)$ coincides with the number of irreducible components of $\mathcal{C}(n, c)$.*

Proof. Clearly, the number of irreducible components of $\mathcal{C}(n, c)$ is the same as the number of irreducible components of $\mathcal{V}(n, c) := \mathcal{C}(n, c) \times \text{Hom}(W, V)$. Let $\mathcal{V}_1(n, c), \dots, \mathcal{V}_p(n, c)$ denote the irreducible components of $\mathcal{V}(n, c)$, and set $\mathcal{V}_l(n, c)^{\text{st}} := \mathcal{V}_l(n, c) \cap \mathcal{V}(n, c)^{\text{st}}$, with $l = 1, \dots, p$. Since for every $(B_0, \dots, B_{n-1}) \in \mathcal{C}(n, c)$ there is $I \in \text{Hom}(W, V)$ such that (B_0, \dots, B_{n-1}, I) is stable, it follows that $\mathcal{V}_l(n, c)^{\text{st}} \neq \emptyset$ for each $l = 1, \dots, p$.

Moreover, since the group $G := GL(V)$ is irreducible, it is easy to see that if $x \in \mathcal{V}_l(n, c)^{\text{st}}$ then its orbit $G \cdot x \subset \mathcal{V}_l(n, c)^{\text{st}}$. Note also that $\mathcal{V}_l(n, c) //_{\chi} G = \mathcal{V}_l(n, c)^{\text{st}} / G$ is irreducible, for each $l = 1, \dots, p$.

Since the GIT quotient $\mathcal{M}(n, c)$ coincides, by Proposition 2.7, with the set of stable G -orbits, we have that

$$\mathcal{M}(n, c) = (\mathcal{V}_1(n, c)^{\text{st}} / G) \cup \dots \cup (\mathcal{V}_p(n, c)^{\text{st}} / G)$$

and the desired conclusion follows from Corollary 7.4. \square

As an immediate consequence of the results on the irreducibility of the variety of commuting matrices mentioned above, we obtain the following new facts regarding the irreducibility of $\text{Hilb}^{[c]}(\mathbb{C}^n)$.

Corollary 9.2.

- (i) $\text{Hilb}^{[c]}(\mathbb{C}^3)$ is irreducible for $c \leq 10$ and reducible for $c \geq 30$;
- (ii) for $n \geq 4$, $\text{Hilb}^{[c]}(\mathbb{C}^n)$ is irreducible if and only if $c \leq 3$.

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